STABILITY OF PERIODIC EXCITATIONS FOR ONE MODEL OF "SONIC VACUUM"

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The modulation stability of the periodic solution for one model of "sonic vacuum," which is a system of elastic granules interacting by the Hertz law, is proved by Whitham's method. The ratio between the phase velocity of an envelope wave and the appropriate sound velocity is found.

In [1, 2] a nonlinear equation describing wave propagation in a one-dimensional chain of spherical granules was obtained in the long-wave approximation. A partial exact periodic solution of this equation was found in explicit form. It has been shown that under certain conditions there is also a qualitatively new soliton-like solution, which is a carrying tone rather than sound excitation for this system. In [3] similar results were obtained for a whole class of analogous essentially nonlinear systems. A characteristic feature of such media is that the long wave sound velocity at zero initial deformation is equal to zero. This was the basis for introducing the concept of "sonic vacuum" in [3] to distinguish this class of systems. Note that these new soliton-like solutions were obtained in the experiments of [2] as well. Other examples of "sonic vacuum" realized in practice are presented in [4]. In the present work, on the basis of the variational principle of the corresponding long-wave approximation for a system of granules, the modulation stability of the exact periodic solution is shown. It is proved that the corresponding system of equations of modulation is hyperbolic.

In the long-wave approximation the equation describing the waves in a one-dimensional chain of spherical granules takes the form [1, 2]

$$u_{tt} = c^2 \left\{ \frac{3}{2} (-u_x)^{1/2} u_{xx} + \frac{a^2}{8} (-u_x)^{1/2} u_{xxxx} - \frac{a^2 u_{xx} u_{xxx}}{8 (-u_x)^{1/2}} - \frac{a^2}{64} \frac{u_{xx}^3}{(-u_x)^{3/2}} \right\}. \tag{1}$$

Equation (1) may also be written in the divergent form

$$u_{tt} = c^2 \left\{ (-u_x)^{3/2} + \frac{a^2}{32} \left[(-u_x)^{-1/2} u_{xx}^2 - 4 (-u_x)^{1/2} u_{xxx} \right] \right\}_x. \tag{2}$$

Here $a = 2R$, R is the granule radius, $c^2 = 2E/(\pi\rho_0(1 - \nu^2))$, E is Young's modulus, ν is Poisson's ratio, ρ_0 is the density of the granule material, and u_x is the deformation.

Equations (1) and (2) are the Euler equation for the variational principle

$$\delta \iint L dt dx = 0,$$

where

$$L = \frac{u_t^2}{2} - \frac{c^2 (-u_x)^{5/2}}{5/2} - \frac{c^2 a^2}{2} \left\{ \frac{3}{8} (-u_x)^{1/2} u_{xx}^2 - \frac{1}{3} (-u_x)^{3/2} u_{xxx} \right\}. \tag{3}$$

Following Whitham [5], let us seek the solution of Eqs. (1) and (2) in the form

$$u = \frac{\Psi(T, X)}{\varepsilon} + \varphi(\theta, T, X) + \dots \tag{4}$$

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Here, points denote terms of higher order in ε; $T = \varepsilon t$, $X = \varepsilon x$ are the slow variables; $\theta = \Theta(T, X)/\varepsilon$ is the phase; ε is a small parameter, which is the ratio of the length of the periodic wave to that of the modulation wave. The functions Ψ, φ and Θ are to be determined. The quantities $k = \Theta_X$ and $\omega = -\Theta_T$ are called the local wave number and frequency. Substituting Eq. (4) into (3) in the zeroth approximation, we obtain the Lagrangian

$$L_0 = \frac{(\Psi_T - \omega\varphi_\theta)^2}{2} - \frac{c^2(-\Psi_X - k\varphi_\theta)^{5/2}}{5/2} - \\ - \frac{c^2a^2}{2}\left\{\frac{3}{8}(-\Psi_X - k\varphi_\theta)^{1/2}k^4\varphi_{\theta\theta}^2 - \frac{1}{3}(-\Psi_X - k\varphi_\theta)^{3/2}k^3\varphi_{\theta\theta\theta}\right\}. \tag{5}$$

Let us introduce the following notation:

$$\xi = -\Psi_X - k\varphi_\theta, \quad V = \frac{\omega}{k}. \tag{6}$$

Then (5) will take the form

$$L_0 = \frac{(\Psi_T + V(\Psi_X + \xi))^2}{2} - \frac{c^2\xi^{5/2}}{5/2} - \frac{c^2a^2k^2}{2}\left\{\frac{3}{8}\xi^{1/2}\xi_\theta^2 + \frac{1}{3}\xi^{3/2}\xi_{\theta\theta\theta}\right\}. \tag{7}$$

The dependence of ξ on θ is determined by varying (5) with respect to φ

$$\frac{V^2}{c^2}\xi_\theta = \frac{3}{2}\xi^{1/2}\xi_\theta + \frac{a^2k^2}{8}\left(\xi^{-1/2}(\xi\xi_{\theta\theta})_\theta - \frac{\xi_\theta^3\xi^{-3/2}}{8}\right). \tag{8}$$

As is shown in [1, 2], the equation has the exact partial solution

$$\xi = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^2 \cos^4\left(\frac{\sqrt{10}}{5ak}\theta\right), \tag{9}$$

which vanishes to zero at discrete values of θ. Formally, Eq. (8) has a singularity in these points. However, direct substitution may prove that there is no singularity indeed. The stability of solution (9) will be studied. Let us introduce the averaging operator

$$\langle\cdot\rangle = \frac{1}{\tau}\oint \cdot\, d\theta,$$

where integration by period is done; τ is the period. Then, from (6) and (7) it follows that

$$\langle\xi\rangle = -\Psi_X,$$

$$\langle L_0\rangle = \frac{\Psi_T^2 - V^2\Psi_X^2 + V^2\langle\xi^2\rangle}{2} - \frac{c^2\langle\xi^{5/2}\rangle}{5/2} - \frac{c^2a^2k^2}{2}\left\{\frac{3}{8}\langle\xi^{1/2}\xi_\theta^2\rangle + \frac{1}{3}\langle\xi^{3/2}\xi_{\theta\theta\theta}\rangle\right\}.$$

Since

$$\frac{1}{\tau}\oint \cos^{2n}\left(\frac{\sqrt{10}}{5ak}\theta\right)d\theta = \frac{1}{\pi}\oint \cos^{2n} w\,dw = \frac{(2n-1)!!}{2^n n!},$$

it follows from (9) that

$$\langle\xi\rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^2\frac{3}{8} = -\Psi_X,$$

$$\langle\xi^2\rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^4\frac{7!!}{2^4\cdot 4!}, \quad \langle\xi^{5/2}\rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^5\frac{9!!}{2^5\cdot 5!},$$

$$\langle \xi^{1/2}\xi_\theta^2 \rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^5 \frac{32}{5a^2k^2}\langle \cos^8 w - \cos^{10} w\rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^5 \frac{32}{5a^2k^2}\left(\frac{7!!}{2^4\cdot 4!} - \frac{9!!}{2^5\cdot 5!}\right),$$

$$\langle \xi^{3/2}\xi_{\theta\theta\theta} \rangle = \left(\frac{5}{4}\frac{V^2}{c^2}\right)^5 \left(\frac{\sqrt{10}}{5ak}\right)^3 8\,\langle (3\cos^7 w - 8\cos^9 w)\sin w\rangle = 0.$$

Since

$$\frac{5}{4}\frac{V^2}{c^2} = \sqrt{-\frac{8}{3}\Psi_X},$$

then

$$\langle L_0 \rangle = \frac{\Psi_T^2}{2} - \frac{10\sqrt{2}}{9\sqrt{3}}c^2(-\Psi_X)^{5/2}. \qquad (10)$$

The Lagrangian $\langle L_0 \rangle$ (10) averaged on solution (9) is analogous to the first two terms of the Lagrangian L [Eq. (3)]. Only the coefficient is different, with

$$\frac{10\sqrt{2}}{9\sqrt{3}} > \frac{2}{5}. \qquad (11)$$

Thus, for Ψ we have the equation

$$\Psi_{TT} + \frac{25}{9}\frac{\sqrt{2}}{\sqrt{3}}c^2((-\Psi_X)^{3/2})_X = 0, \qquad (12)$$

which is equivalent to the equations of the one-dimensional theory of elasticity or, which is the same, to the equations of motion of a barotropic gas written in Lagrangian mass coordinates.

In the system (12) a "gradient catastrophe" is probable, which may be overcome if allowance is made for the corrections of higher order in ε. The corresponding equation is likely to be analogous to (1). In this case, one can expect the solitons of envelopes to appear.

In (12) the value $-\psi_X$ plays the role of the local phase velocity of the envelope wave with deformation c_1:

$$c_1^2 = \frac{25}{3\sqrt{6}}(-\Psi_X)^{1/2}c^2. \qquad (13)$$

It is interesting to compare the ratio of c_1 to the other characteristic velocity values in the present system, for example, to long-wave sound speed c_0 at the same deformation $-\psi_X$. For c_0 we have the expression [1, 2]

$$c_0^2 = \frac{3}{2}(-\Psi_X)^{1/2}c^2. \qquad (14)$$

From (13) and (14) it is easily seen that by virtue of (11) the following relationship is valid:

$$c_1\,[(-\Psi_X)] > c_0\,[(-\Psi_X)].$$

The hyperbolic character of Eq. (12) implies the stability of solution (9). Note that the corresponding gas dynamics analogy was also found in describing rapidly oscillating solutions of the equations of a bubbly liquid with an incompressible liquid phase at oscillations close to resonance ones [6].

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